On the Ultraviolet Behaviour of Newton's constant

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Abstract. We clarify a point concerning the ultraviolet behaviour of the Quantum Field Theory of gravity, under the assumption of the existence of an ultraviolet Fixed Point. We explain why Newton's constant should to scale like the inverse of the square of the cutoff, even though it is technically inessential. As a consequence of this behaviour, the existence of an UV Fixed Point would seem to imply that gravity has a built-in UV cutoff when described in Planck units, but not necessarily in other units.

It has often been speculated that spacetime is intrinsically grainy at very small scales, leading to a cutoff on momentum integrations. The most convincing arguments in favor of this idea probably come from recent results in Loop Quantum Gravity (for a review see [1] and references therein), which predicts for example that the area has a discrete spectrum with a minimum non-zero eigenvalue. One would like to be able to reconcile this view with the continuum, functional methods of Quantum Field Theory (QFT) that have been so successfully applied to other interactions.

We do not know any QFT of gravity that is both unitary and perturbatively renormalizable, so the best hope for a consistent QFT of gravity seems to lie in some form of nonperturbative renormalizability. This could be realized by the existence of a nontrivial fixed point (FP) for the Renormalization Group (RG), as suggested in [2] where the notion of asymptotic safety was introduced. Recently, some significant evidence in support of this hypothesis has appeared in the literature [3, 4, 5, 6]. Let us recall briefly the features of asymptotic safety that are relevant here. A QFT can be defined by an infinite set of coupling constants multiplying all possible operators compatible with the symmetries of the theory. In order for the theory to be considered "fundamental", as opposed to an "effective field theory" that holds only up to a given cutoff, it must yield finite numerical values for all physical processes at all energies. Now, not all the couplings will be involved in the final result of the calculation of a physical quantity. In particular, a coupling that can be eliminated from the action by a pointwise redefinition of the fields can also be eliminated from the formula for a physical quantity by the same redefinition. Such couplings are called "inessential" in [2]. The wave function renormalization constant Z is a typical example of an inessential coupling, since it can be set to one through a constant rescaling of the field. If one calculates a reaction rate within this theory and expresses it by means of such rescaled variables, one will get a result independent of Z. In order for the theory to be finite one must impose that the reaction rate be finite at all energies. Using dimensional arguments, this implies that the dimensionless essential couplings (that is, the essential couplings divided by suitable powers of the energy scale, so that the ratio is dimensionless) must go to a constant when the energy tends to infinity. Equivalently, the corresponding beta functions have to go to zero. Since Z (as any other inessential coupling) does not appear explicitly in the reaction rates, no contraint needs to be imposed about its behaviour at infinite energy.

In this note we do not give any new evidence for or against the existence of a FP. Rather, we discuss what the properties of a FP for gravity should be *if* the theory possesses it. In particular, we clarify a point that has so far remained somewhat obscure in discussions of asymptotic safety, namely the behaviour of Newton's constant at a FP. Our arguments do not rely on any particular assumption about the form of the running effective action. The point is that, independently of the form of the action, one can make Newton's constant disappear by a constant rescaling of the metric. This means exactly that Newton's constant is inessential: in fact, it is in some sense the wave function renormalization constant of the graviton. According to the previous discussion, then,

it would seem that one need not impose any condition on the asymptotic behaviour of Newton's constant. We discuss why, due to the peculiar character of gravity, this expectation is incorrect and Newton's constant must scale like the square of the inverse of the cutoff at a gravitational FP. Our discussion should be considered as a prequel to the work in [3, 4, 5, 6], where this behaviour is indeed observed in a QFT of gravity, within specific classes of actions.

Before discussing gravity, it is useful to review first the way in which the RG works in an ordinary QFT in flat space. We begin by assuming that physics at a certain energy scale k can be accurately described by a local effective action Γ_k containing a finite number of terms, with coupling constants g_i. One is interested in calculating the effective action at some lower energy scale k/a with $a > 1$. This can be done by performing a functional integration over fluctuations of the fields with momenta q in the range $k/a < |q| < k$ ‡, using Γ_k as the bare action in the path integral. In general, this functional integral will produce infinitely many effective couplings, but under suitable conditions the result can again be well approximated by an action of the same form, with the only difference that the coupling constants g_i have different values. Taking the limit $a \to 1$ one can compute the beta functions $\beta_i = \partial_t g_i$, where $t = \log(k/k_0)$, for some arbitrary k_0. These beta functions depend in general on all couplings g_j. Integrating the corresponding first order differential equations produces one-parameter families of effective actions Γ_k, all of the same form but with running coupling constants. In some cases the theory breaks down at some scale; it is then to be regarded just as an effective field theory. In others, it is possible to take the limit $k \to \infty$; one can then regard the theory as being "fundamental".

There are two technical points to be taken into account at this stage. First, dimensionful quantities do not have a measurable value: one can only measure their ratio to some other quantity of the same dimension. Therefore, in the RG, it is customary to take the cutoff k as a unit of mass and to consider the beta functions of the dimensionless ratios $\tilde{g}_i = g_i/k^{d_i}$, where d_i is the mass dimension of the coupling g_i.

Second, as mentioned earlier, one can eliminate the inessential couplings from the action by means of field redefinitions. For example in the case of the wave-function renormalization constant Z one can always rescale the field (and the remaining couplings) in such a way that $Z = 1$. The functional integral over an infinitesimal momentum shell however modifies the value of Z, so if we want to maintain $Z = 1$ at all scales, it is necessary to rescale the fields at every RG step.

An infinitesimal RG transformation therefore consists of the following three steps [7]: 1) an integration over an infinitesimal shell of momenta from k to $k - dk$ (*i.e.* $a = 1 + \delta t$); 2) a rescaling of all momenta by a factor $1 - \delta t$ so that k is restored as a unit of momentum; 3) a redefinition of the fields to restore the values of the inessential couplings.

Since we are interested in applications of this formalism in a gravitational context,

‡ We consider a Euclidean theory.

it is more convenient to view the second step as a rescaling of the metric. This is equivalent to a rescaling of coordinates and momenta in the case of a flat metric, but generalizes also to curved metrics. Let us illustrate this point in the specific case of a single scalar field ϕ with an action of the form

$$S(\phi, g_{\mu\nu}; Z_\phi, m^2, \lambda, \ldots) = \int d^4x\sqrt{g}\left[\frac{Z_\phi}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2 + \lambda\phi^4 + \ldots\right] , \qquad (1)$$

where the ellipses stand for other terms with higher powers of the field and/or higher derivatives. (We are not interested here in the issue whether this theory actually has a FP, but only in the methodology.) The metric plays the role of an external field and can be assumed to be flat. We assume the coordinates to be dimensionless and $g_{\mu\nu}$ to have dimension of length squared. This action is invariant under the following two-parameter family of rescalings:

$$S(\phi, g_{\mu\nu}; Z_\phi, m^2, \lambda, \ldots) = S(bc\phi, b^{-2}g_{\mu\nu}; c^{-2}Z_\phi, b^2c^{-2}m^2, c^{-4}\lambda, \ldots) . \qquad (2)$$

The power of c in front of a coupling counts the power of ϕ appearing in the corresponding term of the action. The power of b in front of a coupling counts twice the power of $g_{\mu\nu}$ minus the power of ϕ appearing in the corresponding term in the action. Note also that the power of b in front of any quantity is equal to its canonical dimension. In fact this *is* the definition of the canonical dimension of every quantity appearing in eq. (1).

If one scales all quantities, including the cutoff, in the appropriate way, invariance under scalings can be extended to the quantum theory. Assuming that it can be approximated by an action of the form (1), the running effective action Γ_k can be shown to have the invariance

$$\Gamma_k(\phi, g_{\mu\nu}; Z_\phi, m^2, \lambda, \ldots) = \Gamma_{bk}(bc\phi, b^{-2}g_{\mu\nu}; c^{-2}Z_\phi, b^2c^{-2}m^2, c^{-4}\lambda, \ldots) , \qquad (3)$$

with b, c real numbers. This can be generalized to arbitrarily complicated actions. Note that the cutoff behaves just like another dimensionful coupling. One can use the c-invariance to fix for example $Z_\phi = 1$ and the b-invariance to fix $k = 1$. One can then define

$$\tilde\Gamma(\tilde\phi, \tilde g_{\mu\nu}, \tilde m^2, \tilde\lambda, \ldots) = \Gamma_1(\tilde\phi, \tilde g_{\mu\nu}, 1, \tilde m^2, \tilde\lambda, \ldots) = \Gamma_k(\phi, g_{\mu\nu}, Z_\phi, m^2, \lambda, \ldots) , \qquad (4)$$

where $\tilde\phi = \sqrt{Z_\phi}\phi/k$, $\tilde m^2 = m^2/(k^2 Z_\phi)$, $\tilde\lambda = \lambda/Z_\phi^2$, $\tilde g_{\mu\nu} = k^2 g_{\mu\nu}$.

With these definitions, the second step in a RG transformation is a scaling with a parameter $b = 1 - \delta t$, which brings k back to its original value, and the third step is a scaling with a parameter $c = 1 - \frac{1}{2}\delta t\eta_\phi$, with $\eta_\phi = \frac{\partial_t Z_\phi}{Z_\phi}$ which brings Z_ϕ back to its original value. These transformations produce a flow for the action $\tilde\Gamma$.

When one computes the beta functions of the essential couplings, they can be written as functions of the essential couplings and of the anomalous dimension η_ϕ. There is no explicit dependence on Z_ϕ or k. A FP for the scalar field would be a point

where the beta functions of the essential couplings vanish. The beta function of Z_ϕ is not required to vanish; instead, the anomalous dimension η_ϕ at the FP can be computed once the values of the essential couplings at the FP are known.

Up to a point, things work in the same way also for gravity. There are no obstacles to performing quantum calculations in gravity as long as one regards it as an effective field theory [8]. One can also write exact, nonperturbative RG equations for gravity: since the range of momenta to be integrated over in a RG step is finite, there are no divergences in the definition of the beta functions. By means of suitable approximations, one can thus obtain equations that describe the evolution of the gravitational couplings for finite k [9, 10]. The issue is whether it is possible to consistently let k go to infinity.

For the sake of definiteness consider pure gravity with the following action:

$$S(g_{\mu\nu};\Lambda,Z_g,\ldots)=\int d^4x\sqrt{g}\,[2Z_g\Lambda-Z_gR[g]+\ldots]\ , \tag{5}$$

where $Z_g=(16\pi G)^{-1}$, Λ is the dimension-two cosmological constant and the dots stand for higher powers of curvature. As discussed earlier, one should treat separately the essential and inessential gravitational couplings. The coupling Z_g has the same role as the wave function renormalization Z_ϕ. In the linearized theory, it multiplies the kinetic term of the graviton, and it can be entirely eliminated from the action by a rescaling of the metric§ and therefore is an inessential coupling‖.

The action (5) has the following scaling property:

$$S(g_{\mu\nu};\Lambda,Z_g,\ldots)=S(b^{-2}g_{\mu\nu};b^2\Lambda,b^2Z_g,\ldots)\ . \tag{6}$$

As with scalar fields, this turns into a scaling property of the running effective action:

$$\Gamma_k(g_{\mu\nu};\Lambda,Z_g,\ldots)=\Gamma_{bk}(b^{-2}g_{\mu\nu};b^2\Lambda,b^2Z_g,\ldots)\ . \tag{7}$$

Following the example of the scalar field, we can use this freedom to set $k=1$. We define

$$\tilde\Gamma(\tilde g_{\mu\nu};\tilde Z_g,\tilde\Lambda,\ldots)=\Gamma_1(\tilde g_{\mu\nu};\tilde Z_g,\tilde\Lambda,\ldots)=\Gamma_k(g_{\mu\nu};Z_g,\Lambda,\ldots)\ , \tag{8}$$

where

$$\tilde g_{\mu\nu}=k^2g_{\mu\nu}\ ;\quad \tilde Z_g=\frac{Z_g}{k^2}=\frac{1}{16\pi\tilde G}\ ;\quad \tilde\Lambda=\frac{\Lambda}{k^2}\ . \tag{9}$$

If this were any other QFT, it would also be possible to eliminate the inessential coupling Z_g. But here the peculiar character of gravity emerges. In the case of the scalar field (or any other field theory), the scaling of the field (parameter c in eq. (3)) and the scaling of the momenta (parameter b in eq. (3)) are two independent operations. In the case of gravity, since the length of any vector (in particular the modulus of the momentum)

§ In the linearized theory, the background and the fluctuation have to be rescaled in the same way.
‖ It has been argued that perhaps Newton's constant is essential because of boundary terms in the gravitational action [11]. One can avoid this type of arguments assuming that the fields fall off sufficiently fast at infinity.

is determined by $g_{\mu\nu}$, the scaling of the field and the scaling of the momenta are the same operation. This is why in eqs. (6–7) we have only a one-parameter scaling, while in (2–3) we had a two-parameter scaling. As a result of this fact, in the case of gravity it is impossible to eliminate at the same time k and Z_g from the action.

If we choose to eliminate k as in (8), the effective action and also the beta functions of all couplings will depend on Z_g (or equivalently on Newton's constant G). As a consequence, at a FP, one has to require the beta function of $\tilde{G}$ to vanish, along with those of all the essential couplings.

There is also an alternative procedure: one can use the b-freedom to set $Z_g = Z'_g$ for some arbitrary fixed value Z'_g. Then we can define a new action depending only on Λ' and the other essential couplings:

$$\Gamma'_{k'}(g'_{\mu\nu};\Lambda',\ldots) = \Gamma_{k'}(g'_{\mu\nu};\Lambda',Z'_g,\ldots) = \Gamma_k(g_{\mu\nu};\Lambda,Z_g,\ldots)\ , \tag{10}$$

where

$$g'_{\mu\nu} = \frac{Z_g}{Z'_g} g_{\mu\nu}\ ;\ \Lambda' = \frac{Z'_g}{Z_g}\Lambda\ ;\ k' = \sqrt{\frac{Z'_g}{Z_g}}k\ . \tag{11}$$

There does not seem to be any reason to assume that b is dimensionless, therefore one can choose $Z'_g = 1/16\pi$, which amounts to working in Planck units.

If we choose to eliminate Z_g as in equation (10), the beta functions for the essential couplings will retain an explicit dependence on the variable k'. Reexpressing the derivatives in terms of the independent variable $t' = \log k'$:

$$\partial_{t'}\Lambda' = \frac{2}{2-\eta}\partial_t\Lambda' = \beta_{\Lambda'}(\Lambda',k',\ldots), \tag{12}$$

where $\eta = \partial_t Z_g/Z_g$ and the ellipses stand for all the essential couplings of the theory. This behaviour can indeed be verified for the beta functions given for example in [9, 10], and is most clear in those cases when the cutoff is chosen in such a way that the momentum integrals can be performed explicitly, as for example in [12].

Thus, if we choose to work in Planck units we obtain a non-autonomous system of RG flow equations. This is unlike any other field theory, where the beta functions never depend explicitly on the cutoff. The explicit k'-dependence makes the very notion of FP a priori somewhat unclear, since the evolution of the couplings has to follow a time-dependent vectorfield: a point where the beta function vanishes instantaneosly is not a FP in general because the zero moves.

In practice, the simplest way to solve for the flow defined by eq. (12) is to exploit the relations between the variables and map the flow of $\tilde{\Lambda}$ and $\tilde{G}$ as a function of t onto the flow of Λ' as a function of t'. Referring now to the explicit calculations in [3, 4, 5, 6, 13], when one actually solves for the flow, the picture is slightly more complicated due to the fact that the flow of $\tilde{\Lambda}$ and $\tilde{G}$ near the FP follows a spiralling motion. Thus, k' does not grow monotonically with k: it first overshoots its FP-value k'_* and then approaches it with damped oscillations. As a consequence, k' is not a good independent variable for describing the flow of Λ' (it can only be used on finite intervals where $\frac{dk'}{dk} \neq 0$).

This problem can be circumvented by using a different independent variable, that can be constructed as follows. In a suitable coordinate system in a neighborhood of the FP in the $\tilde{\Lambda}$-$\tilde{G}$ plane, the linearized vectorfield is given by $\partial_t g_i(t) = M_{ij} g_j(t)$, where $M_{11} = M_{22} = -\alpha$ and $M_{21} = -M_{12} = \omega$, with $\alpha \approx 1.86$ and $\omega \approx 4.08$. There is no real linear transformation that diagonalizes M. In polar coordinates ρ, φ the solutions of this equation are $\rho(t) = \rho_0\, e^{-\alpha t}$, $\varphi(t) = \varphi_0 + \omega t$.

One can perform a diffeomorphism of the $\tilde{\Lambda}$-$\tilde{G}$ plane which undoes the spiralling motion, such that the new $\tilde{G}$ variable is a monotonic function of t. We cannot write the explicit form of this diffeomorphism, but in the neighborhood of the FP it is $\rho' = \rho$ and $\varphi' = \varphi + \frac{\omega}{\alpha} \log(\rho)$. The transformation is singular at the FP and can be joined smoothly to the identity far from the FP. Denoting with y and x the Cartesian coordinates corresponding to the polar coordinates ρ' and φ', the flow equations become simply $\frac{dx}{dt} = -\alpha x$ and $\frac{dy}{dt} = -\alpha y$. For $t \to -\infty$, $x \to \tilde{G}$ and $y \to \tilde{\Lambda}$. Except for the line $x = 0$, one can now take x as the new independent variable, in which case eq. (10) takes the simple form $\frac{dy}{dx} = \frac{y}{x}$. The trajectories of the class Ia and IIa in the terminology of [13] lie in the region $x < 0$ and have $\frac{dx}{dt} > 0$, while those of the class IVa lie in the region $x > 0$ and have $\frac{dx}{dt} < 0$.

Let us now return to the more familiar picture in cutoff units. The fact that Newton's constant behaves in some sense like an essential coupling, has the following consequences. On one hand, in spite of not being an essential parameter, $\tilde{G}$ has to reach a finite limit $\tilde{G}_*$ at an UV FP. Now we observe that $k' = \sqrt{\tilde{G}}$, *i.e.* the cutoff in Planck units is (the square root of) Newton's constant in cutoff units. Consequently, if gravity had a FP, the cutoff would have a finite limit

$$k'_* = \sqrt{\tilde{G}_*} \tag{13}$$

in Planck units.

On the other hand, since

$$\eta = \frac{\partial_t \tilde{Z}_g}{\tilde{Z}_g} + 2\ , \tag{14}$$

the anomalous dimension η has to be exactly equal to 2 at a non-Gaussian FP [4], suggesting that at very small scales the world may look two-dimensional.

When matter is added to the picture additional possibilities arise, giving rise to interesting scenarios [14]. We found in [5] that adding a scalar field with interactions of the form ϕ^{2n} and $\phi^{2n}R$ does not alter significantly the result of pure gravity, in the following sense: under the same approximations that produce a FP for pure gravity, there is still a FP, with slightly shifted values of Λ_* and G_* and all scalar couplings with $n \geq 1$ equal to zero.

Now, ordinary systems of units are based on atomic spectroscopy. The scale of the energy levels of atoms is set by the mass of the electron, which is in turn dictated by the VEV of the Higgs field. In this sense, ordinary units are based on the mass of the Higgs field. If the system admits a FP, the results of [5] suggest that the ratio v^2/k^2

goes to zero at this FP, v^2 being the VEV of the scalar field. Therefore, if we interpret ϕ as the modulus of the Higgs field, there would be no minimal length in Higgs units. On the other hand (13) can be taken as an indication that the theory has a minimal length in Planck units. These remarks show that the answer to the question whether gravity has a minimal length could depend upon the system of units.

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